Non-linear dynamics, entanglement and the quantum-classical crossover of two coupled SQUID rings

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Received Day Month Year

Revised Day Month Year

We explore the quantum-classical crossover of two coupled, identical, superconducting quantum interference device (SQUID) rings. We note that the motivation for this work is based on a study of a similar system comprising two coupled Duffing oscillators. In that work we showed that the entanglement characteristics of chaotic and periodic (entrained) solutions differed significantly and that in the classical limit entanglement was preserved only in the chaotic-like solutions. However, Duffing oscillators are a highly idealised toy model. Motivated by a wish to explore more experimentally realisable systems we now extend our work to an analysis of two coupled SQUID rings. We observe some differences in behaviour between the system that is based on SQUID rings rather than on Duffing oscillators. However, we show that the two systems share a common feature. That is, even when the SQUID ring's trajectories appear to follow (semi) classical orbits entanglement persists.

Keywords: SQUID's, entanglement, decoherence.

1. Introduction

In this paper we report the results of an initial study into the entanglement properties associated with the quantum classical crossover of two coupled SQUID rings. This work can be considered as being part of a much larger study spanning many years (see, for example, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17). The overarching theme being the quantum-classical crossover of systems the exhibit non-linear dynamics in their classical motion. Our interest arises from the fact the Schrödinger equation is linear. Hence, when applying the correspondence principle, how can one recover classical like trajectories of classically non-linear systems in terms of expectation values of quantum mechanical observables? A general solution appears to take the form of introducing environmental degrees of freedom together with a suitable expression of the correspondence principle.

This particular study was motivated by a previous investigation[14] of two coupled duffing oscillators. In that work we wished to extend the analysis of the quantum classical crossover of a single Duffing oscillator[1,2,3,4,5,6,9,10] to more complex systems. Specifically we wished to study the the recovery of classical like orbits from a quantum description of two coupled identical Duffing oscillators. Here our interest lay in understanding how the purely quantum mechanical phenomena of entanglement would change as the coupled

system approached the classical limit. We showed *"that the entanglement characteristics of two classical states (chaotic and periodic solutions) differ significantly in the classical limit. In particular, we show[ed] that significant levels of entanglement are preserved only in the chaotic-like solutions"*[14]. Although the Duffing oscillator has been studied for many years with great interest it is, nevertheless, a toy model. Hence, recently, our focus has moved to include the study of SQUID rings. These are devices that can manifest similar properties as the Duffing oscillator (such as a double well potential and classical, dissipative, chaos) but are also readily fabricated. Hence, in this paper we investigate the entanglement characteristics for the quantum-classical crossover of a system comprising of two identical coupled SQUID rings.

The correspondence principle in quantum mechanics is usually expressed in the form: *"For those quantum systems with a classical analogue, as Planck's constant becomes vanishingly small the expectation values of observables behave like their classical counterparts"*[18]. Applying this expression of the correspondence principle together with introduction of environmental degrees of freedom via unravelling the master equation is the standard method of recovering trajectories of classically non-linear systems. Indeed, we used this procedure to recover classical like trajectories for our work on two coupled Duffing oscillators. However, in 17 we found that this formulation of the correspondence principle cannot be applied to SQUID rings. Instead we proposed a more pragmatic version as follows: Consider $\hbar$ fixed (it is) and scale the Hamiltonian so that when compared with the minimum area $\hbar/2$ in phase space:

(a) the relative motion of the expectation values of the observable become large and
(b) the state vector is localised.

Then, under these circumstances, expectation values of operators will behave like their classical counterparts[17].

In this work the the evolution of the state vector $|\psi\rangle$ of the open system comprising of two coupled SQUID rings subject to Ohmic damping will be modelled using a quantum state diffusion unravelling of the master equation. This model takes the form of an Itô increment equation given by[12,13]

$$
\begin{aligned}
|d\psi\rangle = &-\frac{i}{\hbar}\hat{H}_{sys}\,|\psi\rangle\,dt \\
&+\sum_j\left[\left\langle\hat{L}_j^\dagger\right\rangle\hat{L}_j-\frac{1}{2}\hat{L}_j^\dagger\hat{L}_j-\frac{1}{2}\left\langle\hat{L}_j^\dagger\right\rangle\left\langle\hat{L}_j\right\rangle\right]|\psi\rangle\,dt \\
&+\sum_j\left[\hat{L}_j-\left\langle\hat{L}_j\right\rangle\right]|\psi\rangle\,d\xi
\end{aligned}
\tag{1}
$$

where the first term on the right hand side of this equation is the Schrödinger evolution of the system with Hamiltonian $\hat{H}_{sys}$. The effect of the environment is modelled by the addition of a second (drift) and third (fluctuation) term that describe the decohering effects of the environment on the systems state vector. In this work we consider only one Lindblad operator per subspace $\hat{L}_j = \sqrt{2\zeta}\hat{a}_j$, where a_j is annihilation operator. The time increment

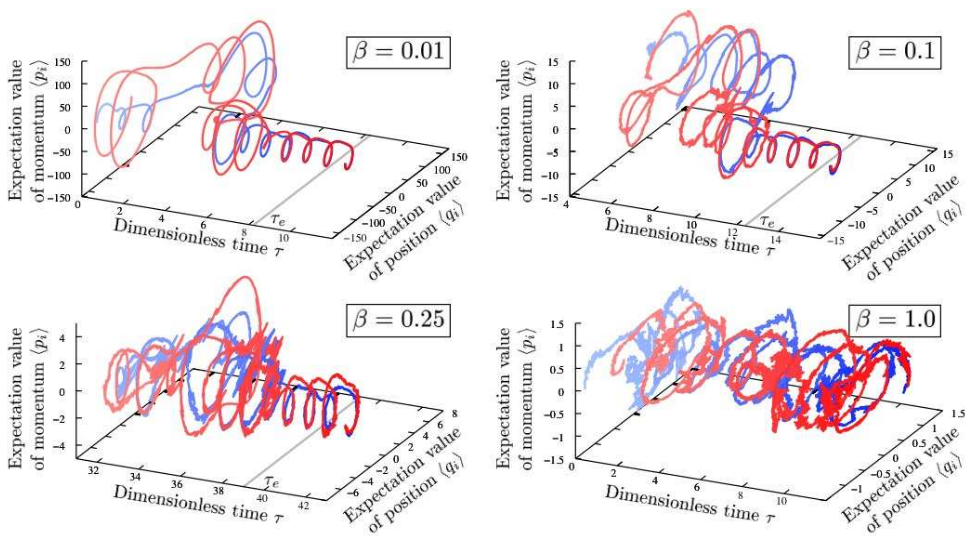

Fig. 1. The dynamics of the expectation values of position and momentum as a function of time (normalised to drive periods) for $\beta = 0.01$, $\beta = 0.1$, $\beta = 0.25$ and $\beta = 1.0$. The dynamics have been taken over the same duration and displayed so that, with the exception $\beta = 1.0$, they approximately align at the time (τ_e) at which entrainment occurs. (NB: Figure and caption reproduced from 14)

is dt and the $d\xi$ are complex Weiner increments satisfying $\overline{d\xi^2} = \overline{d\xi} = 0$ and $\overline{d\xi d\xi^*} = dt$[12,13] where the over-bar denotes the average over infinitely many stochastic processes.

2. Review of the entanglement properties of two coupled Duffing oscillator

In order to set the scene for our new results we now summarise some of the text and results from our earlier work on entanglement dynamics in coupled Duffing oscillators[14]. In that work we extended the analysis of previous work on such systems[10,5] and considered two identical, coupled Duffing oscillators. The Hamiltonian for each oscillator was given by

$$H_i = \frac{1}{2}p_i^2 + \frac{\beta^2}{4}q_i^4 - \frac{1}{2}q_i^2 + \frac{g_i}{\beta}\cos(t)\, q_i + \frac{\Gamma_i}{2}(q_i p_i + p_i q_i) \tag{2}$$

where q_i and p_i are the position and momentum operators for each oscillator. As usual for Ohmic damping, the Lindblad operators were simply $L_i = \sqrt{2\Gamma_i}a_i$ (for $i = 1, 2$), where a_i is the annihilation operator. As with previous work in this field[10,5] in 14 we set the parameters $g_i = 0.3$ and $\Gamma_i = 0.125$ respectively. The Hamiltonian for the coupled system then was taken to be:

$$H = H_1 + H_2 + \mu q_1 q_2 \tag{3}$$

where we set the coupling strength $\mu = 0.2$. In this work the parameter β is a scaling parameter for $\hbar$ with $\beta = 1$ being the fully classical limit.

Solutions to the equation of motion for the state vector (1) are shown in Figure 1. Here we show the evolution of the expectation values of position and momentum for the two

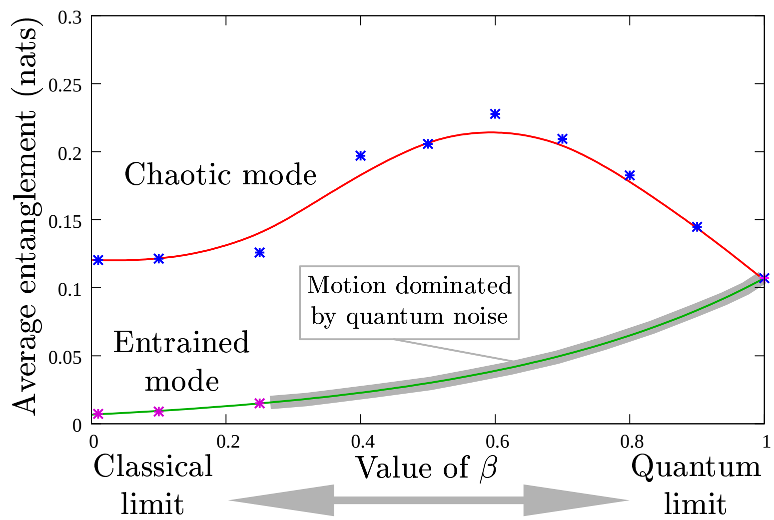

Fig. 2. Mean entanglement entropy as a function of β for the chaotic-like and periodic (entrained) states. Here we see that the entanglement entropy for system in the chaotic state does not vanish as β approaches the classical regime. (NB: Figure and caption reproduced from 14)

coupled oscillators as a function of time for a four different values of β. The emergence of classical trajectories for small β is clearly apparent. Furthermore for $\beta = 0.01, 0.1$ and 0.25 chaotic behaviour of the ceases after some time and the motion becomes almost periodic.

We measured the entanglement between the two oscillators by computing the entropy of entanglement for the system[19]. That is the von Neumann entropy[20] of the reduced density operator ρ_i for one of the oscillators i.e.

$$S\left(\rho_i\right) = -\mathrm{Tr}[\rho_i \ln \rho_i].$$

We then demonstrated that that when these coupled Duffing oscillators enter the almost periodic entrained state the entanglement falls rapidly as the system approaches the classical regime. However, when both oscillators undergo chaotic like motion we found that significant average entangled remained even when the dynamics of the system appear to be classical. We found these results (shown in figure 2) surprising, as entanglement, which does not have a classical counterpart, persisted even for classical looking trajectories.

3. RSJ model and scaling the dynamics

As already noted, in 17 we found that the SQUID ring Hamiltonian was incompatible with the correspondence principle phrased in terms of making Planck's constant vanishingly small. A solution to this problem was found by scaling the system so that the underlying dynamics remain, qualitatively, the same. We can see how this may be achieved for SQUID rings by looking at the classical equations of motion that are given by the resistively shunted junction (RSJ) model. The RSJ equation of motion for the magnetic flux, Φ, within a driven

SQUID ring is[21]:

$$C\frac{d^2\Phi}{dt^2}+\frac{1}{R}\frac{d\Phi}{dt}+\frac{\Phi-\Phi_x}{L}+I_c\sin\left(\frac{2\pi\Phi}{\Phi_0}\right)=I_d\sin\left(\omega_d t\right) \tag{4}$$

where Φ_x, C, I_c, L and R are, respectively, the external flux bias, capacitance and critical current of the weak link, the inductance of the ring and the resistance. The drive amplitude and frequency are I_d and ω_d respectively and $\Phi_0 = h/2e$ is the flux quantum.

We can then rewrite (4) in the standard, universal oscillator like, form by making the following definitions: $\omega_0 = 1/\sqrt{LC}$, $\tau = \omega_0 t$, $\varphi = (\Phi - \Phi_x)/\Phi_0$, $\varphi_x = \Phi_x/\Phi_0$, $\beta = 2\pi L I_c/\Phi_0$, $\omega = \omega_d/\omega_0$, $\varphi_d = I_d L/\Phi_0$ and $\zeta = 1/2\omega_0 RC$. This yields the following equation of motion:

$$\frac{d^2\varphi}{d\tau^2}+2\zeta\frac{d\varphi}{d\tau}+\varphi+\frac{\beta}{2\pi}\sin\left[2\pi\left(\varphi+\varphi_x\right)\right]=\varphi_d\sin\left(\omega\tau\right) \tag{5}$$

In this system of units we then see that we can scale the system Hamiltonian through changing either $C \to aC$ or $L \to bL$ so long as we also make the following changes: $R \to \sqrt{b/a}R$, $I_d \to I_d/\sqrt{b}$ and $\omega_d \to \omega_d/\sqrt{ab}$.

In this paper we use the following basic circuit parameters $C = 1\times10^{-13}$F, $L = 3\times10^{-10}$H, $R = 100\Omega$, $\beta = 2$, $\omega_d = \omega_0$ and $I_d = 0.9\,\mu$A. We note that we have biased the ring at the half flux quantum, $\Phi_x = 0.5\Phi_0$, so that the potential approximates a double well. We change a so that C varies between 1×10^{-16} F (quantum limit) and 1×10^{-9} F (classical limit), changing other circuit parameters in line with the above methodology.

4. Quantum mechanical description of the SQUID Ring

We now turn to the quantum description of this system. The SQUID ring Hamiltonian for each of the identical rings on their own is[22]:

$$\hat{H}_i=\frac{\hat{Q}_i^2}{2C}+\frac{\left(\hat{\Phi}_i-\Phi_{x_i}(t)\right)^2}{2L}-\frac{\hbar I_c}{2e}\cos\left(\frac{2\pi\hat{\Phi}_i}{\Phi_0}\right) \tag{6}$$

where the magnetic flux threading each ring, $\hat{\Phi}_i$, and the total charge across each weak link $\hat{Q}_i$ take on the roles of conjugate variables for the system with the imposed commutation relation $\left[\hat{\Phi}_i,\hat{Q}_i\right]=i\hbar$ and $\Phi_0 = h/2e$ is the flux quantum. Here $\Phi_{x_i}(t)$ is the external applied magnetic flux and incorporates the drive term for each ring.

We define dimensionless flux and charge operators in the manner usual for the simple harmonic oscillator: $\hat{x}_i = \sqrt{C\omega_0/\hbar}\hat{\Phi}_i$ and $\hat{p}_i = \sqrt{1/\hbar C\omega_0}\hat{Q}_i$.

$\hat{H}_i' = \hat{H}_i/\hbar\omega_0$ we find that

$$\hat{H}_i'=\frac{\hat{p}_i^2}{2}+\frac{\left[\hat{x}_i-x_i(t)\right]^2}{2}-\frac{I_c}{2e\omega_0}\cos\left(\Omega\hat{x}_i\right) \tag{7}$$

where $\Omega=\left[(4e^2/\hbar)\sqrt{(L/C)}\right]^{1/2}$.

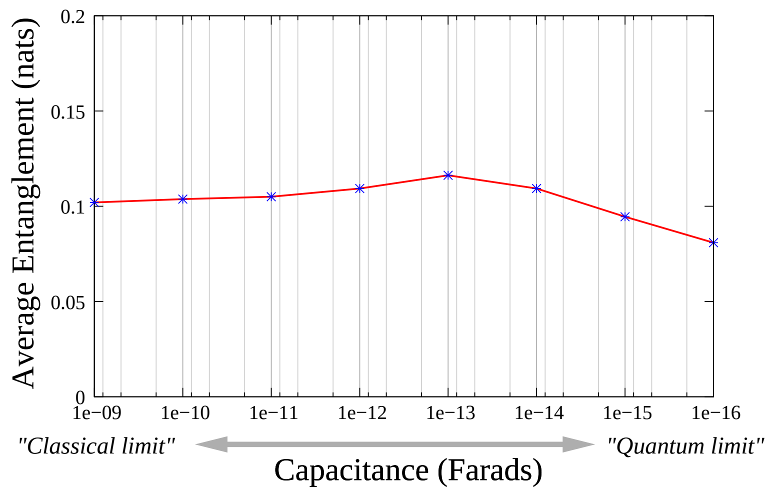

Fig. 3. Mean entanglement entropy as a function of Capacitance two coupled SQUID rings. Here we see that the entanglement entropy for system does not vanish even as it approaches its classical limit.

We note that this method of introducing Ohmic damping does not bring with it the frequency shift that arises through the damping term in the classical dynamics. We resolve this problem, as for the Duffing oscillator, by the addition of an extra term to the Hamiltonian[18,5]. That is, the Hamiltonian

$$\hat{H}_i' = \frac{\hat{p}_i^2}{2} + \frac{[\hat{x}_i - x_i(t)]^2}{2} - \frac{I_c}{2e\omega_0}\cos(\Omega\hat{x}_i) + \frac{\zeta}{2}(\hat{p}_i\hat{x}_i + \hat{x}_i\hat{p}_i) \tag{8}$$

is one we can use to generate trajectories that will be comparable to those predicted by the RSJ model.

$$\hat{H}_{total} = \sum_{i\in\{1,2\}} \left\{ \frac{\hat{p}_i^2}{2} + \frac{[\hat{x}_i - x_i(t)]^2}{2} - \frac{I_c}{2e\omega_0}\cos(\Omega\hat{x}_i) + \frac{\zeta}{2}(\hat{p}_i\hat{x}_i + \hat{x}_i\hat{p}_i) \right\} + \mu\hat{x}_1\hat{x}_2$$

where $\mu = 0.2$

5. Results and comparison with the Duffing oscillator

We are now in a position to present results for two coupled SQUID rings for comparison with figure 2. Here we have computed the average entanglement as a function of capacitance (for a selection of scale parameters a). However we note that the entanglement entropies presented here are is the average entanglement over either a long time period or many similar trajectories. It is not the entanglement associated with the average density operator taken of many experiments. This average entanglement cannot therefore be considered usable in a quantum information sense. In figure 3 we show this average entanglement entropy. Here the averaging of each trajectory was determined on a point by point

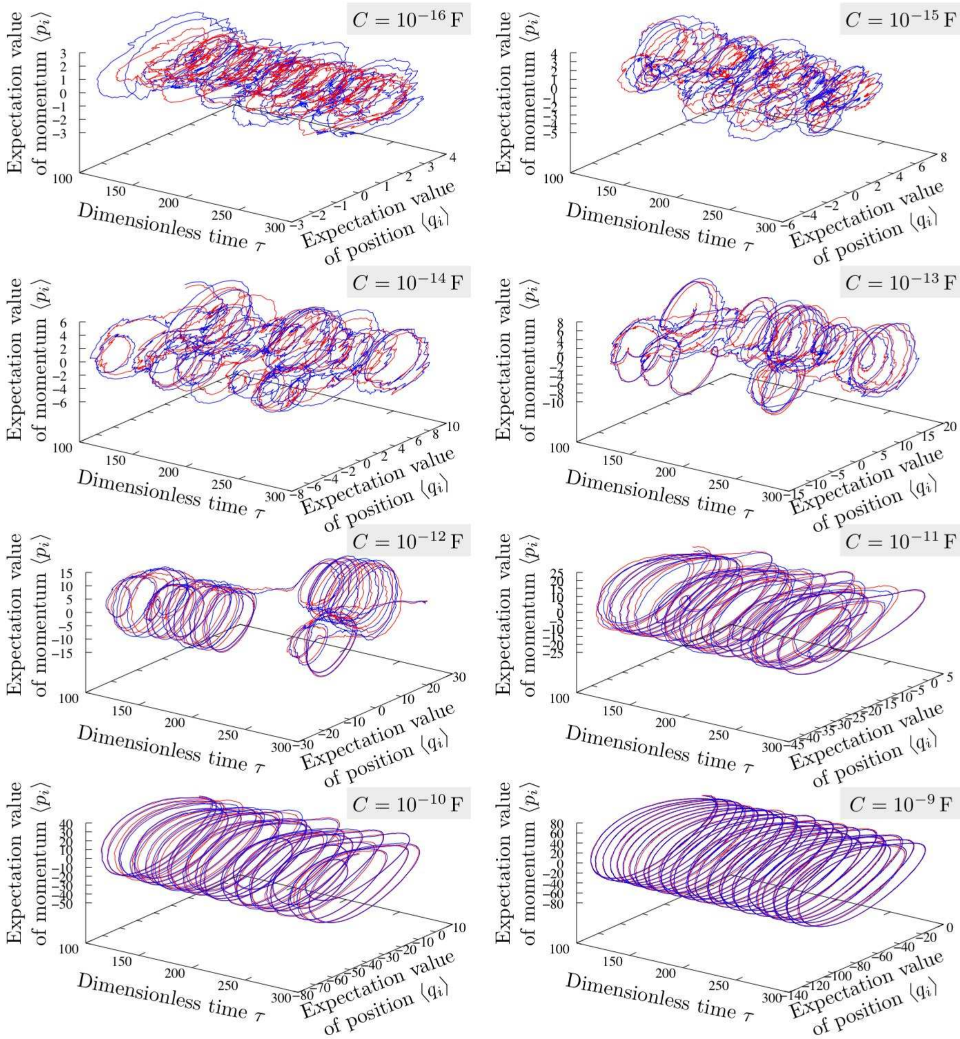

Fig. 4. The dynamics of the expectation values of normalised flux and charge as a function of dimensionless time for a range of capacitances.

basis. A sufficient averaging was used so as to ensure that the results presented here had settled to within a percent or so.

As for the Duffing oscillators, here the mean entanglement does not appear to vanish in the classical limit (large capacitance). Another surprising feature in common with the Duffing oscillator results is that the average entropy is not maximum at the most quantum limit (smallest capacitance).

From our previous study of the Duffing oscillator as well as that presented in 23 we expect this average entanglement of the pure state to be associated with and underlying

chaotic or non-linear dynamics. In figure 4 we show sample phase portraits for the expectation values of flux and charge as a function of dimensionless time. For very small capacitance's we see the motion is dominated by noise fluctuations. As the capacitance increases we see, as expected, the emergence of non-linear dynamical behaviour. However rather surprisingly we see that for large values of capacitance (i.e. the classical limit) this non-linear dynamics gives way to a nearly periodic motion. This result for SQUID rings is in stark contrast to those for the system comprising of Duffing oscillators. Whilst we find these results most interesting we are, here, reporting our preliminary results in the study of coupled SQUID rings in the correspondence limit. We will explore these results in more detail in later work.

6. Conclusion

We have shown that for two coupled SQUID rings the average entanglement persists as the rings approach their classical limit. Indeed the trend seems very similar to that associated with the chaotic motion of two coupled Duffing oscillators. We note that for the uncoupled rings the dynamics remains non-linear for all values of the scale parameter a. However, we have seen this is not the case for the coupled system. Unlike the entanglement entropies for two coupled Duffing oscillators with the SQUID ring based system we see no sudden change in the mean entanglement entropy associated with this change in the underlying dynamics. Specifically, even in nearly periodic orbits the SQUID rings exhibit non-negligible entanglement entropies even when the underlying dynamics appears almost classical and periodic. As this entanglement is associated with nearly periodic motion it may form the basis for extracting usable entanglement from this system. This is something that we will explore in a more detailed investigation of this system at a later date.

Acknowledgments

The author would like to thank The Physics Grid (Loughborough) and Loughborough University HPC service for use of their facilities.

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